Ergodic BSDEs under weak dissipative assumptions

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Abstract

In this paper we study ergodic backward stochastic differential equations (EBSDEs) dropping the strong dissipativity assumption needed in [12]. In other words we do not need to require the uniform exponential decay of the difference of two solutions of the underlying forward equation, which, on the contrary, is assumed to be non degenerate.

We show existence of solutions by use of coupling estimates for a non-degenerate forward stochastic differential equations with bounded measurable non-linearity. Moreover we prove uniqueness of "Markovian" solutions exploiting the recurrence of the same class of forward equations.

Applications are then given to the optimal ergodic control of stochastic partial differential equations and to the associated ergodic Hamilton-Jacobi-Bellman equations.

1 Introduction

Since the beginning of the 90's several papers have described the link between backward stochastic differential equations (BSDEs), Hamilton-Jacobi-Bellman equations and stochastic optimal control (see, for instance, [8] and [19]). The successive literature on BSDEs covered several different situations, among them infinite horizon control problems, both in finite and infinite dimensions (see [3], [21] and [14]).

In [12] the BSDE approach was extended to the case of ergodic control problems, that is, of control problems in which the cost functional only evaluates the long time behavior of the stochastic system. In that paper the authors introduced the following class of BSDEs with infinite horizon, called Ergodic BSDEs (EBSDEs):

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_\sigma^x, Z_\sigma^x) - \lambda] d\sigma - \int_t^T Z_\sigma^x dW_\sigma, \quad \mathbb{P}-\text{a.s.}, \quad \forall\, 0 \le t \le T < \infty, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a cylindrical Wiener process in a Hilbert space Ξ, X^x is the solution of the following forward SDE with values in a Hilbert (in [12] also Banach) space H,

$$dX_t^x = (AX_t^x + F(X_t^x))dt + GdW_t, \quad X_0 = x, \tag{1.2}$$

and $\psi : H \times \Xi^* \to \mathbb{R}$ is a given function.

We underline that the unknowns in the above equation is the triple (Y, Z, λ), where Y, Z are adapted processes taking values in $\mathbb{R}$ and Ξ^*, respectively, and λ is a real number.

The case of the EBSDEs driven by a finite dimensional reflected forward equation together with its applications to semilinear PDEs with general Neumann boundary conditions was then treated in [20].

The main assumption in [12] (and, with slight modifications, in [20]) is the strict dissipativity of $A+F$, i.e., the existence of $k>0$ such that for all x and x' in the domain of A, the following holds

$$(A(x-x') + F(x) - F(x'), x-x') \leq -k|x-x'|_H^2.$$

Such a requirement ensures the uniform exponential decay of the difference between the trajectories of two solutions of equation (1.2) that plays a crucial role in the arguments in [12].

The aim of the present paper is to show that, when G is invertible, we can drop the dissipativity assumption on $A+F$ and study (1.1) when A is dissipative but F is only assumed to be bounded and Lipschitz with no restrictions on its Lipschitz constant. See Example 5.1.1 to compare, in the concrete case of an ergodic problem for a stochastic heat equation, the assumptions needed in the present paper and the ones needed in [12] .

Our main tool is a coupling estimate for a perturbed version of the forward stochastic differential equation (1.2). Coupling estimates have been recently developed for many different classes of stochastic partial differential equations and exploited to deduce regularity properties of the corresponding Markov semigroup, see e.g. [7], [15], [16], [22] and [18]. In the present paper, in comparison with the previously mentioned literature, we are only dealing with bounded and everywhere defined non-linearities but we have to consider measurable non-linearities and to prove that the estimate depends on them only through their sup (see Theorem 2.4 and Appendix 6.1).

The coupling estimate is used here to get a uniform bound for $Y_t^{x,\alpha} - Y_0^{0,\alpha}$ where $Y^{x,\alpha}$ is the solution of the following strictly monotonic BSDE (see [21])

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T (\psi(X_\sigma^x, Z_\sigma^{x,\alpha}) - \alpha Y_\sigma^{x,\alpha})d\sigma - \int_t^T Z_\sigma^{x,\alpha} dW_\sigma, \quad 0 \leq t \leq T < \infty.$$

Then the Bismut-Elworthy formula for BSDEs (see [10]) yields the uniform bound for $\nabla_x Y_0^{x,\alpha}$ that allows to pass to the limit as $\alpha \searrow 0$ in the above equation. Note that the non degeneracy of the noise, i.e., the invertibility of G, is used in an essential way at this step. It is also used to prove the coupling estimate but a more sophisticated coupling argument, which would not need this assumption, could be used.

We also notice that we construct a "Markovian" solution of the EBSDE in the sense that Y_t and Z_t are deterministic functions of X_t^x. We prove, by use of the recurrence of the perturbed forward stochastic differential equation, that such a "Markovian" solution of the EBSDE is unique. The recurrence property is studied in [6] for a forward SDE similar to ours; the difference is that here we need to consider drifts that are only bounded and measurable. (see Theorem 2.6 and Appendix 6.2). The uniqueness argument is inspired by the corresponding one in [13].

Once existence and uniqueness of a Markovian solution of the EBSDE is proved we can proceed as in [12] to deal with an optimal control problem with state equation

$$dX_t^{x,u} = (AX_t^{x,u} + F(X_t^{x,u}) + GR(u_t))dt + GdW_t, \quad X_0^{x,u} = x, \tag{1.3}$$

and ergodic cost functional

$$J(x,u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{u,T} \int_0^T L(X_s^x, u_s)ds. \tag{1.4}$$

Then we deduce that the ergodic Hamilton Jacobi Bellman equation

$$\mathcal{L}v(x) + \psi\,(x, \nabla v(x)G) = \lambda, \quad x \in E, \tag{1.5}$$

has a unique mild solution; moreover the ergodic problem admits a unique optimal control that satisfies an optimal feedback law given in terms of the gradient on the solution to the HJB equation (1.5); finally the optimal cost is λ.

In the finite dimensional case there are several papers devoted to the study, by analytic techniques, of stochastic optimal ergodic control problems and of the corresponding HJB equations (see for instance [1] and [2]). On the contrary, to the best of our knowledge, there are very few works devoted to the infinite dimensional case. As far as we know (mild) solutions of an equation like (1.5) was studied, in the infinite dimensional case, only in [13] (besides the already discussed results included in [12]).

In [13] authors prove, by a fixed point argument, existence and uniqueness of the solution of the mild stationary HJB equation for discounted infinite horizon costs. Then they pass to the limit, as the discount goes to zero. They work under the same non-degeneracy assumption that we use here and assume that A is the generator of a contraction semigroup and F is dissipative; they also have a limitation on the Lipschitz constant (with respect to the gradient variable) of the Hamiltonian function ψ (see [11] for similar conditions in the case of a strictly monotonic stationary HJB equation). On the contrary unbounded non-linearities F can be considered in [13].

The present paper is organized as follows. First we establish general notation. In section 2, we introduce the forward equation and state the coupling estimates and recurrence property for the perturbed forward equation. The ergodic BSDE is studied in Section 3. Sections 4 and 5 shortly recall how the previous results can be applied to the ergodic Hamilton-Jacobi-Bellman equation and to the ergodic optimal control problem. We include the proofs for the coupling estimates and the recurrence property for the perturbed forward equation in the Appendix (see section 6).

1.1 General Notation

We introduce some notations; let E, F be real separable Hilbert spaces. The norms and the scalar product will be denoted $|\cdot|$, $\langle\cdot\,,\cdot\rangle$, with subscripts if needed. $L(E,F)$ is the space of linear bounded operators $E \to F$, with the operator norm. The domain of a linear (unbounded) operator A is denoted $D(A)$.

Given $\phi \in B_b(E)$, the space of bounded and measurable functions $\phi : E \to \mathbb{R}$, we denote $\|\phi\|_0 = \sup_{x\in E}|\phi(x)|$. If, in addition, ϕ is also Lipschitz continuous then $\|\phi\|_{\text{lip}} = \|\phi\|_0 + \sup_{x,x'\in E,\, x\neq x'} |\phi(x) - \phi(x')||x - x'|^{-1}$.

We say that a function $F : E \to F$ belongs to the class $\mathcal{G}^1(E,F)$ if it is continuous, has a Gateaux differential $\nabla F(x) \in L(E,F)$ at any point $x \in E$, and for every $k \in E$ the mapping

$x \to \nabla F(x)k$ is continuous from E to F (i.e. $x \to \nabla F(x)$ is continuous from E to $L(E,F)$ if the latter space is endowed with the strong operator topology). In connection with stochastic equations, the space $\mathcal{G}^1$ has been introduced in [9], to which we refer the reader for further properties.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ we consider the following classes of stochastic processes with values in a real separable Banach space K.

1. $L^p_{\mathcal{P}}(\Omega, C([0,T],K))$, $p \in [1,\infty)$, $T > 0$, is the space of predictable processes Y with continuous paths on $[0,T]$ such that
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega,C([0,T],K))} = \mathbb{E} \sup_{t\in[0,T]} |Y_t|^p_K < \infty.$$
2. $L^p_{\mathcal{P}}(\Omega, L^2([0,T];K))$, $p \in [1,\infty)$, $T > 0$, is the space of predictable processes Y on $[0,T]$ such that
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega,L^2([0,T];K))} = \mathbb{E} \left(\int_0^T |Y_t|^2_K \, dt \right)^{p/2} < \infty.$$
3. $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;K))$ is the space of predictable processes Y on $[0,\infty)$ that belong to the space $L^2_{\mathcal{P}}(\Omega, L^2([0,T];K))$ for every $T > 0$.

2 The forward SDE

2.1 General assumptions

This section is devoted to the following mild Itô stochastic differential equation for an unknown process X_τ, $\tau \in \mathbb{R}^+$, with values in a Hilbert space H:
$$\hat{X}_\tau = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}\Upsilon(\hat{X}_\sigma)\, d\sigma + \int_t^\tau e^{(\tau-\sigma)A} G\, dW_\sigma, \qquad \forall \tau \geq 0, \quad \mathbb{P}-\text{a.s.}. \tag{2.1}$$
We assume the following:

Hypothesis 2.1

(i) *A is an unbounded operator $A : D(A) \subset H \to H$, with $D(A)$ dense in H. We assume that A is dissipative and generates a stable C_0-semigroup $\{e^{tA}\}_{t\geq 0}$. By this we mean that there exist constants $k > 0$ and $M > 0$ such that*
$$\langle Ax, x\rangle \leq -k|x|^2 \quad \forall x \in D(A); \qquad |e^{\tau A}| \leq Me^{-k\tau}.$$

(ii) *For all $s > 0$, e^{sA} is a Hilbert-Schmidt operator. Moreover $|e^{sA}|_{L_2(H,H)} \leq L\, s^{-\gamma}$ for suitable constants $L > 0$ and $\gamma \in [0, 1/2)$.*

(iii) *Υ is a bounded measurable map $H \to H$,*

(iv) *G is a bounded linear operator in $L(\Xi, H)$. Moreover we assume that G is invertible and we denote by G^{-1} its bounded inverse.*

(v) *$(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(\mathcal{F}_t)_{t\geq 0}$ is a filtration in it satisfying the usual conditions and $(W_t)_{t\geq 0}$ is an $\mathcal{F}$-cylindrical Wiener process with values in a separable Hilbert space Ξ.*

Remark 2.2 We notice that if the operator A with dense domain is m-dissipative that is $\langle Ax, x\rangle \le -k|x|_H^2 \quad \forall x \in D(A)$ and $A - k_1 I$ is surjective for a suitable $k_1 > 0$ then by the Lumer-Phillips theorem it follows immediately that A generates stable C_0-semigroup of contractions (that is $M = 1$).

The following result is well known in its first part (see, for instance, [5]) and a straight-forward consequence of the Girsanov transform in the second.

Proposition 2.3 *Fix $t \ge 0$ and $x \in H$ and assume that Υ is Lipschitz. Under the assumptions of Hypothesis 2.1 there exists a unique adapted process $\hat{X}$ verifying (2.1). Moreover, for every $p \in [2, \infty)$ and every $T > t$, $\hat{X} \in L^p_{\mathcal{P}}(\Omega; C([0,T]; H))$ and*

$$\mathbb{E} \sup_{\tau \in [t,T]} |X_\tau|^p \le C(1 + |x|)^p, \tag{2.2}$$

for some constant C depending only on p, γ, M and $\sup_{x\in H} |\Upsilon(x)|$ but independent of $T > t$.

If Υ is only bounded and measurable, then the solution to equation (2.1) still exists but in weak *sense. By this we mean, see again [5], that there exists a new $\mathcal{F}$-Wiener process $(\hat{W}_t)_{t\ge 0}$ with respect to a new probability $\hat{\mathbb{P}}$ (absolutely continuous with respect to $\mathbb{P}$), and an $\mathcal{F}$-adapted process $\hat{X}$ with continuous trajectories for which (2.1) holds with W replaced by $\hat{W}$. Moreover (2.2) still holds (with respect to the new probability). Finally such a weak solution is unique in law.*

In the following we will denote the solution of equation (2.1) by $\hat{X}^{t,x}$ and by $\hat{X}^x$ when we choose the initial time $t = 0$. We remark that equation (2.1) is the mild version of the Cauchy problem:

$$\begin{cases} d\hat{X}^{t,x}_s = A\hat{X}^{t,x}_s ds + \Upsilon(\hat{X}^{t,x}_s)dt + GdW_s, \qquad s \ge t, \\ \hat{X}^{t,x}_t = x. \end{cases} \tag{2.3}$$

The following result is proved in section 6.1.

Theorem 2.4 (Basic coupling estimate) *Assume that $\Upsilon : H \to H$ is Lipschitz and let $\hat{X}^x$ be the (strong) solution of equation (2.1) then there exist $\hat{c} > 0$ and $\hat{\eta} > 0$ such that for all $\phi \in B_b(H)$*

$$\big|\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](x')\big| \le \hat{c}(1 + |x|^2 + |x'|^2)e^{-\hat{\eta}t}|\phi|_0, \tag{2.4}$$

where $\mathcal{P}_t[\phi](x) = \mathbb{E}\phi(\hat{X}^x_t)$ is the Kolmogorov semigroup associated to equation (2.1).

We stress the fact that $\hat{c}$ and $\hat{\eta}$ depend on Υ only through $\sup_{x\in H} |\Upsilon(x)|$.

Corollary 2.5 *Relation (2.4) can be extended to the case in which Υ is only bounded and measurable and there exists a uniformly bounded sequence of Lipschitz functions $\{\Upsilon_n\}_{n\ge 1}$ (i.e. $\forall n, \Upsilon_n$ is Lipschitz and $\sup_n \sup_x |\Upsilon_n(x)| < \infty$) such that*

$$\lim_n \Upsilon_n(x) = \Upsilon(x), \quad \forall x \in H.$$

Clearly in this case in the definition of $\mathcal{P}_t[\phi]$ the mean value is taken with respect to the new probability $\hat{P}$.

Proof. It is enough to show that if $\mathcal{P}^n$ is the semigroup corresponding to equation (2.1) with Υ replaced by Υ_n, then $\forall x \in H$ and $\forall t \ge 0$,

$$\mathcal{P}^n_t[\phi](x) \to \mathcal{P}_t[\phi](x).$$

We set

$$U_\tau^x = e^{\tau A}x + \int_0^\tau e^{(\tau-\sigma)A} G\, dW_\sigma.$$

By Girsanov's formula

$$\mathcal{P}_t^n[\phi](x) = \mathbb{E}(\rho_t^{n,x}\phi(U_t^x)), \qquad \mathcal{P}_t[\phi](x) = \mathbb{E}(\rho_t^x \phi(U_t^x))$$

where

$$\rho_t^{n,x} = \exp\left(-\int_0^t \langle G^{-1}\Upsilon_n(U_t^x), dW_s\rangle_\Xi - \frac{1}{2}\int_0^t |G^{-1}\Upsilon_n(U_t^x)|_\Xi^2 ds\right),$$

and

$$\rho_t^x = \exp\left(-\int_0^t \langle G^{-1}\Upsilon(U_t^x), dW_s\rangle_\Xi - \frac{1}{2}\int_0^t |G^{-1}\Upsilon(U_t^x)|_\Xi^2 ds\right).$$

We have

$$\begin{aligned} & \mathbb{E}[(\rho_t^{n,x})^2] \\ = \ & \mathbb{E}\left[\exp\left(-2\int_0^t \langle G^{-1}\Upsilon_n(U_t^x), dW_s\rangle_\Xi - 2\int_0^t |G^{-1}\Upsilon_n(U_t^x)|_\Xi^2 ds\right)\exp\left(\int_0^t |G^{-1}\Upsilon_n(U_t^x)|_\Xi^2 ds\right)\right] \\ \leq \ & \exp\left(t|G^{-1}|^2 \sup_n \sup_x |\Upsilon_n(x)|^2\right) < \infty, \end{aligned}$$

from which we deduce that $\{\rho_t^{n,x}\}_n$ is uniformly integrable in $L^1(\Omega)$. Moreover, it is easy to see that $\lim_n \rho_t^{n,x} = \rho_t^x$ in probability, and the claim follows. □

The equation (2.1) also enjoys a recurrence property that will be useful in the following, it is proved in section 6.2.

Theorem 2.6 *Assume that $\Upsilon : H \to H$ can be approximated (in the sense of pointwise convergence) by a uniformly bounded sequence of Lipschitz functions $\{\Upsilon_n\}_{n\geq 1}$. Then the solution of equation (2.1) is recurrent in the sense that for all $\Gamma \in H$, Γ open:*

$$\lim_{T\to\infty} \hat{\mathbb{P}}\{\exists t \in [0,T] : \hat{X}_t^x \in \Gamma\} = 1.$$

In particular, setting $\tau^x = \inf\{t : |\hat{X}_t^x| < \epsilon\}$, then $\forall \epsilon > 0$, $\lim_{T\to\infty} \hat{\mathbb{P}}\{\tau^x < T\} = 1$.

3 The Ergodic BSDE

We fix now a bounded function $F : H \to H$ and denote by $X^{t,x}$ (and by X^x when we choose the initial time $t = 0$) the solution of equation (2.1) with $\Upsilon = F$.
This section is devoted to the following type of BSDEs with infinite horizon

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_\sigma^x, Z_\sigma^x) - \lambda]\, d\sigma - \int_t^T Z_\sigma^x\, dW_\sigma, \quad 0 \leq t \leq T < \infty, \tag{3.1}$$

where λ is a real number and is part of the unknowns of the problem; the equation is required to hold for every t and T as indicated. On the function $\psi : H \times \Xi^* \to \mathbb{R}$ and F we assume the following:

Hypothesis 3.1 *ψ is a measurable map $H \times \Xi^* \to \mathbb{R}$. Moreover there exists $l > 0$ such that*

$$|\psi(x,0)| \leq l; \quad |\psi(x,z) - \psi(x,z')| \leq l|z - z'|, \qquad x \in H,\ z, z' \in \Xi^*.$$

Hypothesis 3.2 *F is bounded, Lipschitz and Gâteaux differentiable, more precisely, F belongs to the class $\mathcal{G}^1(H,H)$.*

We start by considering an infinite horizon equation with strictly monotonic drift, namely, for $\alpha > 0$, the equation

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T (\psi(X_\sigma^x, Z_\sigma^{x,\alpha}) - \alpha Y_\sigma^{x,\alpha}) d\sigma - \int_t^T Z_\sigma^{x,\alpha} dW_\sigma, \quad 0 \le t \le T < \infty. \tag{3.2}$$

The existence and uniqueness of solution to (3.2) under Hypothesis 3.1 was first studied by Briand and Hu in [3] and then generalized by Royer in [21]. The following lemma follows from Lemma 2.1 and Theorem 3.2 in [14].

Lemma 3.3 *Let us suppose that the Hypotheses 2.1, 3.1 and 3.2 hold. Then for all $x \in H$ and $\alpha > 0$,*

(i) there exists a unique solution $(Y^{x,\alpha}, Z^{x,\alpha})$ to BSDE (3.2) such that $Y^{x,\alpha}$ is a bounded continuous process, $Z^{x,\alpha}$ belongs to $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$, and $|Y_t^{x,\alpha}| \le l/\alpha$, $\mathbb{P}$-a.s. for all $t \ge 0$;

(ii) if we define $v^\alpha(x) = Y_0^{x,\alpha}$ then, for all fixed $\alpha > 0$, v^α is Lipschitz bounded and of class $\mathcal{G}^1$, moreover,

$$Y_t^{x,\alpha} = v^\alpha(X_t^x), \quad Z_t^{x,\alpha} = \nabla v^\alpha(X_t^x) G.$$

In order to construct the solution to (3.1), we need some uniform in α estimate of $|v^\alpha(x) - v^\alpha(x')|$. This will be obtained by coupling estimates but first we have to prove an approximation lemma:

Lemma 3.4 *Let ζ, $\zeta' : H \to \Xi^*$ weakly* continuous with polynomial growth. We define*

$$\tilde{\Upsilon}(x) = \begin{cases} \dfrac{\psi(x,\zeta(x)) - \psi(x,\zeta'(x))}{|\zeta(x)) - \zeta'(x)|^2} \left(\zeta(x) - \zeta'(x)\right)^*, & \text{if } \zeta(x) \neq \zeta'(x), \\ \\ 0, \ \text{if } \zeta(x) = \zeta'(x). \end{cases}$$

There exists a uniformly bounded sequence of Lipschitz functions $(\tilde{\Upsilon}_n)_{n\ge 1}$ (i.e., $\forall n$, $\tilde{\Upsilon}_n$ is Lipschitz and $\sup_n \sup_x |\tilde{\Upsilon}_n(x)| < \infty$) such that

$$\lim_n \tilde{\Upsilon}_n(x) = \tilde{\Upsilon}(x), \quad \forall x \in H.$$

Proof. Fixing an orthonormal basis $\{\xi_1, \xi_2, \cdots\}$ in Ξ, we define the projection Π_{p,Ξ^*}: $\Xi^* \to \Xi^*$ as follows:

$$\Pi_{p,\Xi^*}\zeta = \sum_{i=1}^{n} (\zeta\xi_i) < \xi_i, \cdot > .$$

Set

$$\widetilde{\Upsilon}^i(x) = \frac{\psi(x,\zeta(x)) - \psi(x,\zeta'(x))}{|\zeta(x) - \zeta'(x)|^2 + i^{-1}} \left(\zeta(x) - \zeta'(x)\right)^*,$$

$$\widetilde{\Upsilon}^{i,p}(x) = \frac{\psi(x,\Pi_{p,\Xi^*}\zeta(x)) - \psi(x,\Pi_{p,\Xi^*}\zeta'(x))}{|\Pi_{p,\Xi^*}\zeta(x)) - \Pi_{p,\Xi^*}\zeta'(x)|^2 + i^{-1}} \left(\Pi_{p,\Xi^*}\zeta(x) - \Pi_{p,\Xi^*}\zeta'(x)\right)^*.$$

It is easy to verify that the functions $\widetilde{\Upsilon}^{i,p}$ are continuous functions. Moreover $|\widetilde{\Upsilon}^{i,p}(x)| \le l$ and $\lim_p \widetilde{\Upsilon}^{i,p}(x) = \widetilde{\Upsilon}^i(x)$, $\lim_i \widetilde{\Upsilon}^i(x) = \widetilde{\Upsilon}(x)$, for all $x \in H$. Fixing i, p, it is quite classical (based on

finite dimensional projections and convolutions) to construct a uniformly bounded sequence of Lipschitz functions $\{\widetilde{\Upsilon}^{i,p,m}\}_m$, such that $\lim_m \widetilde{\Upsilon}^{i,p,m}(x) = \widetilde{\Upsilon}^{i,p}(x)$, see, e.g. Lemma 4.2 in [10]. Then the proof ends with a diagonal procedure. □

The following lemma plays a crucial role. It gives the desired estimate of $v^\alpha(x) - v^\alpha(x')$ and of ∇v^α.

Lemma 3.5 *There exists a constant $c(\ell, \hat{c}, \hat{\eta}) > 0$ such that for all x, $x' \in H$*

$$|v^\alpha(x) - v^\alpha(x')| \leq c(1 + |x|^2 + |x'|^2); \tag{3.3}$$

and for all $x \in H$,

$$|\nabla v^\alpha(x)| \leq c(1 + |x|^2). \tag{3.4}$$

We stress the fact that $c > 0$ is independent of α.

Proof. Set

$$\tilde{\Upsilon}^\alpha(x) = \begin{cases} \dfrac{\psi(x, \nabla v^\alpha(x)G) - \psi(x,0)}{|\nabla v^\alpha(x)G|^2} \left(\nabla v^\alpha(x)G\right)^*, & \text{if } \nabla v^\alpha(x)G \neq 0 \\ \\ 0, \text{ if } \nabla v^\alpha(x)G = 0. \end{cases}$$

Then

$$\psi(X_t^x, Z_t^{x,\alpha}) = \psi(X_t^x, 0) + \tilde{\Upsilon}^\alpha(X_t^x) Z_t^{x,\alpha}.$$

From Proposition 3.4, $\tilde{\Upsilon}^\alpha$ is the pointwise limit of a uniformly bounded sequence of Lipschitz functions.

For all $T > 0$, the couple of processes $(Y^{x,\alpha}, Z^{x,\alpha})$ is a solution to the following finite horizon linear BSDE

$$\begin{cases} -dY_t^{x,\alpha} = \psi(X_t^x, 0)dt + \tilde{\Upsilon}^\alpha(X_t^x) Z_t^{x,\alpha} dt - \alpha Y_t^{x,\alpha} dt - Z_t^{x,\alpha} dW_t, & t \in [0,T], \\ Y_T^{x,\alpha} = v^\alpha(X_T^x). \end{cases} \tag{3.5}$$

Since $\tilde{\Upsilon}^\alpha$ is bounded for all $T > 0$, there exists a unique probability $\hat{\mathbb{P}}^{x,\alpha,T}$ such that

$$\hat{W}_t^{x,\alpha} = \int_0^t \tilde{\Upsilon}^\alpha(X_s^x) ds + W_t$$

is a $\hat{\mathbb{P}}^{x,\alpha,T}$-Wiener process for $t \in [0,T]$. Consequently we have

$$v^\alpha(x) = \hat{\mathbb{E}}^{x,\alpha,T} \left[e^{-\alpha T} v^\alpha(X_T^x) + \int_0^T e^{-\alpha s} \psi(X_s^x, 0) ds \right]$$

where $\hat{\mathbb{E}}^{x,\alpha,T}$ denotes the expectation with respect to $\hat{\mathbb{P}}^{x,\alpha,T}$.

Letting $T \to \infty$, as $|v^\alpha(x)| \leq \frac{l}{\alpha}$, we get

$$v^\alpha(x) = \lim_{T \to \infty} \hat{\mathbb{E}}^{x,\alpha,T} \left[\int_0^T e^{-\alpha s} \psi(X_s^x, 0) ds \right].$$

On the other hand, if we rewrite the forward equation (2.1) with respect to $\hat{W}^{x,\alpha}$ it turns out that X^x verifies

$$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x)dt + G\tilde{\Upsilon}^\alpha(X_t^x)dt + G\hat{W}_t^{x,\alpha}, \\ \hat{X}_0^x = x \in H. \end{cases} \tag{3.6}$$

We denote by $\mathcal{P}^\alpha$ the associated Kolmogorov semigroup, i.e.,

$$\mathcal{P}_t^\alpha[\phi](x) = \hat{\mathbb{E}}^{x,\alpha,t}\phi(X_t^x).$$

Applying Corollary 2.5 with $\Upsilon^\alpha = F + G\tilde{\Upsilon}^\alpha$ (which is also the pointwise limit of a sequence of Lipschitz functions), we obtain

$$|v^\alpha(x) - v^\alpha(x')| \leq \int_0^\infty e^{-\alpha t} \left|\mathcal{P}_t^\alpha[\psi(\cdot,0)](x) - \mathcal{P}_t^\alpha[\psi(\cdot,0)](x')\right| dt \leq \frac{\hat{c}l}{\hat{\eta}}(1 + |x^2| + |x'|^2)$$

where $\hat{c}$ and $\hat{\eta}$ are independent of α. The proof of (3.3) is now complete.

To prove (3.4), let us set

$$\bar{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0).$$

Then, $\bar{Y}_t^{x,\alpha} = Y_t^{x,\alpha} - Y_0^{0,\alpha} = \bar{v}^\alpha(X_t^x)$ is the unique solution of the finite horizon BSDE

$$\begin{cases} -d\bar{Y}_t^{x,\alpha} = \psi(X_t^x, Z_t^{x,\alpha})dt - \alpha\bar{Y}_t^{x,\alpha} - \alpha v^\alpha(0)dt - Z_t^{x,\alpha}dW_t, \\ Y_1^{x,\alpha} = \bar{v}^\alpha(X_1^x). \end{cases}$$

Note that in particular, in the above equation, $|\alpha v^\alpha(0)| \leq l$. By Theorem 4.2 in [10], $\bar{v}^\alpha$ is of class $\mathcal{G}^1$ and there exists a constant $c(l,\hat{c},\hat{\eta}) > 0$ independent of α such that $|\nabla v^\alpha(x)| \leq c(1 + |x|^2)$, and the conclusion follows. □

Remark 3.6 *As already mentioned in the introduction, the non degeneracy assumption on G is essential in the proof of the gradient estimate on v^α. More precisely, it is necessary to use the Bismut-Elworthy formula from [10].*

Now we are in position to state our main result in this section.

Theorem 3.7 *Assume that the Hypotheses 2.1, 3.1 and 3.2 hold. Moreover let $\bar{\lambda}$ be the real number in (3.7) below and define $\bar{Y}_t^x = \bar{v}(X_t^x)$ (where $\overline{v}$ is a locally Lipschitz function with $\overline{v}(0) = 0$ defined in (3.8)). Then there exists a process $\overline{Z}^x \in L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$ such that $\mathbb{P}$-a.s. the EBSDE (3.1) is satisfied by $(\bar{Y}^x, \bar{Z}^x, \bar{\lambda})$ for all $0 \leq t \leq T$.*

Moreover $\overline{v}$ is of class $\mathcal{G}^1$, $|\nabla\overline{v}(x)| \leq c(1 + |x|^2)$, and $\overline{Z}_t^x = \nabla\overline{v}(X_t^x)G$.

Proof. Let us set again $\bar{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0)$. By Lemma 3.3 and relation (3.3) we can construct, by a diagonal procedure, a sequence $\alpha_n \searrow 0$ such that for all x in a countable dense subset $D \subset H$

$$\overline{v}^{\alpha_n}(x) \to \overline{v}(x), \qquad \alpha_n v^{\alpha_n}(0) \to \overline{\lambda}, \tag{3.7}$$

for a suitable function $\overline{v} : D \to \mathbb{R}$ and for a suitable real number $\overline{\lambda}$.

Moreover, by Lemma 3.5, $|\overline{v}^\alpha(x) - \overline{v}^\alpha(x')| \leq c(1 + |x|^2 + |x'|^2)|x - x'|$ for all $x, x' \in H$ and all $\alpha > 0$. So $\overline{v}$ can be extended to a locally Lipschitz function defined on the whole H with $|\overline{v}(x) - \overline{v}(x')| \leq c(1 + |x|^2 + |x'|^2)|x - x'|$ and

$$\overline{v}^{\alpha_n}(x) \to \overline{v}(x), \qquad x \in H. \tag{3.8}$$

Clearly we have, $\mathbb{P}$-a.s.,

$$\overline{Y}_t^{x,\alpha} = \overline{Y}_T^{x,\alpha} + \int_t^T (\psi(X_\sigma^x, Z_\sigma^{x,\alpha}) - \alpha\overline{Y}_\sigma^{x,\alpha} - \alpha v^\alpha(0))d\sigma - \int_t^T Z_\sigma^{x,\alpha}dW_\sigma,\ 0 \leq t \leq T < \infty. \tag{3.9}$$

Since $|\bar{v}^\alpha(x)| \leq c(1+|x|^2)$, inequality (2.2) ensures that $\mathbb{E}\sup_{t\in[0,T]}\left[\sup_{\alpha>0}|\overline{Y}_t^{x,\alpha}|^2\right] < +\infty$ for any $T>0$. Thus, if we define $\overline{Y}^x = \overline{v}(X^x)$, then by dominated convergence theorem

$$\mathbb{E}\int_0^T |\overline{Y}_t^{x,\alpha_n} - \overline{Y}_t^x|^2 dt \to 0 \quad \text{and} \quad \mathbb{E}|\overline{Y}_T^{x,\alpha_n} - \overline{Y}_T^x|^2 \to 0$$

as $n\to\infty$ (where $\alpha_n \searrow 0$ is a sequence for which (3.7) and (3.8) hold).

We claim now that there exists $\overline{Z}^x \in L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$ such that

$$\mathbb{E}\int_0^T |Z_t^{x,\alpha_n} - \overline{Z}_t^x|^2_{\Xi^*} dt \to 0.$$

Let $\tilde{Y} = \bar{Y}^{x,\alpha_n} - \bar{Y}^{x,\alpha_m}$, $\tilde{Z} = Z^{x,\alpha_n} - Z^{x,\alpha_m}$. Applying Itô's rule to $\tilde{Y}^2$ we get, by standard computations, that

$$\tilde{Y}_0^2 + \mathbb{E}\int_0^T |\tilde{Z}_t|^2_{\Xi^*} dt = \mathbb{E}\tilde{Y}_T^2 + 2\mathbb{E}\int_0^T \tilde{\psi}_t \tilde{Y}_t dt - 2\mathbb{E}\int_0^T [\alpha_n Y_t^{x,\alpha_n} - \alpha_m Y_t^{x,\alpha_m}]\, \tilde{Y}_t\, dt,$$

where $\tilde{\psi}_t = \psi(X_t^x, Z_t^{x,\alpha_n}) - \psi(X_t^x, Z_t^{x,\alpha_m})$. We notice that $|\tilde{\psi}_t| \leq l|\tilde{Z}_t|$ and $\alpha_n|Y_t^{x,\alpha_n}| \leq l$. Thus

$$\mathbb{E}\int_0^T |\tilde{Z}_t|^2_{\Xi^*} dt \leq c\left[\mathbb{E}(\tilde{Y}_T)^2 + \mathbb{E}\int_0^T (\tilde{Y}_t)^2 dt + \mathbb{E}\int_0^T |\tilde{Y}_t| dt\right].$$

It follows that the sequence $\{Z^{x,\alpha_n}\}$ is Cauchy in $L^2(\Omega; L^2(0,T;\Xi^*))$ for all $T>0$ and our claim is proved.

Now we can pass to the limit as $n\to\infty$ in equation (3.9) to obtain

$$\overline{Y}_t^x = \overline{Y}_T^x + \int_t^T (\psi(X_\sigma^x, \overline{Z}_\sigma^x) - \overline{\lambda}) d\sigma - \int_t^T \overline{Z}_\sigma^x dW_\sigma, \quad 0 \leq t \leq T < \infty. \tag{3.10}$$

We notice that the above equation also ensures continuity of the trajectories of $\overline{Y}$.

Finally, the couple of processes $(\overline{Y}^x, \overline{Z}^x)$ is the unique solution of the finite horizon BSDE

$$\begin{cases} -d\bar{Y}_t^x = (\psi(X_t^x, \overline{Z}_t^x) - \overline{\lambda})dt - \bar{Z}_t^x dW_t, \\ \bar{Y}_1^x = \bar{v}(X_1^x). \end{cases}$$

Once again, by Theorem 4.2 in [10], we conclude the proof. □

Remark 3.8 The solution we have constructed above has the following "quadratic growth" property with respect to X: there exists $c>0$ such that, $\mathbb{P}$-a.s.,

$$|\overline{Y}_t^x| \leq c(1+|X_t^x|^2), \text{ for all } t \geq 0. \tag{3.11}$$

If we require similar conditions then we immediately obtain uniqueness of λ.

Theorem 3.9 *Assume that the Hypotheses 2.1, 3.1 and 3.2 hold true. Moreover suppose that, for some $x \in H$, the triple (Y', Z', λ') verifies $\mathbb{P}$-a.s. equation (3.1) for all $0 \leq t \leq T$, where Y' is a progressively measurable continuous process, Z' is a process in $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$ and $\lambda' \in \mathbb{R}$. Finally assume that there exists $c_x > 0$ (that may depend on x) such that for some $p \geq 1$, $\mathbb{P}$-a.s.*

$$|Y_t'| \leq c_x(1+|X_t^x|^p), \text{ for all } t \geq 0.$$

Then $\lambda' = \bar{\lambda}$.

The proof of the above theorem is similar to that of Theorem 4.6 in [12], so we omit it here.

The solution obtained in Theorem 3.7 has moreover the property that processes Y^x and Z^x are deterministic functions of X^x. We refer to such solutions as to "Markovian" solution of the EBSDEs.

We prove that the Markovian solution is unique.

Theorem 3.10 *Let (v,ζ), $(\tilde{v},\tilde{\zeta})$ two couples of functions with $v,\tilde{v}: H \to \mathbb{R}$, continuous, with $|v(x)| \le c(1+|x|^2)$, $|\tilde{v}(x)| \le c(1+|x|^2)$, $v(0)=\tilde{v}(0)=0$ and ζ, $\tilde{\zeta}$ continuous from H to Ξ^* endowed with the weak* topology verifying $|\zeta(x)| \le c(1+|x|^2)$, $|\tilde{\zeta}(x)| \le c(1+|x|^2)$.*

Assume that for some constants λ, $\tilde{\lambda}$ and all $x \in H$, $(v(X_t^x),\zeta(X_t^x),\lambda)$, $(\tilde{v}(X_t^x),\tilde{\zeta}(X_t^x),\tilde{\lambda})$ verify the EBSDE (3.1), then $\lambda=\tilde{\lambda}$, $v=\tilde{v}$, $\zeta=\tilde{\zeta}$.

Proof. The equality $\lambda=\tilde{\lambda}$ comes from Theorem 3.9.

Then let $\bar{Y}_t^x = v(X_t^x)-\tilde{v}(X_t^x)$, $\bar{Z}_t^x=\zeta(X_t^x)-\tilde{\zeta}(X_t^x)$ and $\tilde{\Upsilon}$ be defined in Proposition 3.4. We have

$$-d\bar{Y}_t^x = \tilde{\Upsilon}(X_t^x)\bar{Z}_t^x dt - \bar{Z}_t^x dW_t = -\bar{Z}_t^x dW_t'$$

where $W_t' = -\int_0^t \Upsilon(X_s^x)ds + W_t$ is a Wiener process in [0,T] under the probability $\bar{\mathbb{P}}^{x,T}$.

Moreover, under $\bar{\mathbb{P}}^{x,T}$, X^x satisfies equation (2.3), in $[0,T]$, with, as before $\Upsilon = G\widetilde{\Upsilon}+F$. Thus, from Proposition 2.3, it holds that for all $p \ge 1$, and all $x \in H$

$$\bar{\mathbb{E}}^{x,T}|X_t^x|^p \le c(1+|x|^p), \forall 0 \le t \le T,$$

where $c>0$ depends on p,γ,M and $l|G|+\sup_x|F(x)|$, and is independent of T. Thus the growth conditions on ζ and $\tilde{\zeta}$ implies that, for all $T>0$, $\bar{\mathbb{E}}^{x,T}\int_0^T|\bar{Z}_t^x|^2dt<\infty$.

Let $\tau = \inf\{t : |X_t^x|<\epsilon\}$ then for all $T>0$

$$\bar{Y}_0^x = \bar{\mathbb{E}}^{x,T}\bar{Y}_{T\wedge\tau}^x.$$

For any $\delta>0$, there exists $\epsilon>0$ such that $|v(x)-\tilde{v}(x)| \le \delta$ if $|x|\le\epsilon$. Then for a constant $c>0$,

$$\begin{aligned}
|\bar{Y}_0^x| = |\bar{\mathbb{E}}^{x,T}\bar{Y}_{T\wedge\tau}^x| &\le \bar{\mathbb{E}}^{x,T}|\bar{Y}_\tau^x|1_{\{\tau<T\}} + \bar{\mathbb{E}}^{x,T}|\bar{Y}_T^x|1_{\{\tau\ge T\}} \\
&\le \delta + \left(\bar{\mathbb{P}}^{x,T}\{\tau\ge T\}\right)^{1/2}\left(\bar{\mathbb{E}}^{x,T}\{|\bar{Y}_T^x|^2\}\right)^{1/2} \\
&\le \delta + \left(\bar{\mathbb{P}}^{x,T}\{\tau\ge T\}\right)^{1/2}\left(\bar{\mathbb{E}}^{x,T}\{1+|X_T^x|^4\}\right)^{1/2}.
\end{aligned}$$

Noting that, by Theorem 2.6, $\lim_{T\to\infty}\bar{\mathbb{P}}^{x,T}\{\tau\ge T\}=0$ and sending T to ∞ in the last inequality, we obtain that $|\bar{Y}_0^x|\le\delta$ and the claim follows from the arbitrarity of δ. □

4 Ergodic HJB equations

We briefly show here that as $\bar{v}(x)=\bar{Y}_0^x$ in Theorem 3.7 is of class $\mathcal{G}^1$, the couple $(\bar{v},\bar{\lambda})$ is a mild solution of the following "ergodic" Hamilton-Jacobi-Bellman equation:

$$\mathcal{L}v(x)+\psi\left(x,\nabla v(x)G\right)=\lambda, \quad x\in H, \tag{4.1}$$

where the linear operator $\mathcal{L}$ is formally defined by

$$\mathcal{L}f(x) = \frac{1}{2}Trace\left(GG^*\nabla^2 f(x)\right)+\langle Ax,\nabla f(x)\rangle+\langle F(x),\nabla f(x)\rangle.$$

We notice that we can define the transition semigroup $(P_t)_{t\geq 0}$ corresponding to X by the formula $P_t[\phi](x) = E\phi(X_t^x)$ for all measurable functions $\phi : E \to \mathbb{R}$ having polynomial growth, and we notice that $\mathcal{L}$ is the formal generator of $(P_t)_{t\geq 0}$.

Since we are dealing with an elliptic equation it is natural to consider (v, λ) as a mild solution of equation (4.1) if and only if, for arbitrary $T > 0$, $v(x)$ coincides with the mild solution $u(t, x)$ of the corresponding parabolic equation having v as a terminal condition:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathcal{L}u\,(t,x) + \psi\,(x, \nabla u\,(t,x)\,G) - \lambda = 0, \quad t \in [0,T],\ x \in H, \\ \\ u(T,x) = v(x), \quad x \in H. \end{cases} \tag{4.2}$$

Thus we are led to the following definition:

Definition 4.1 *A pair (v, λ) ($v : H \to \mathbb{R}$ and $\lambda \in \mathbb{R}$) is a mild solution of the Hamilton-Jacobi-Bellman equation (4.1) if the following are satisfied:*

1. $v \in \mathcal{G}^1(H, \mathbb{R})$;
2. *there exists $C > 0$ such that $|\nabla v\,(x)| \leq C\,(1 + |x|^p)$ for every $x \in H$ and some $p \geq 1$;*
3. *for $0 \leq t \leq T$ and $x \in H$,*

$$v(x) = P_{T-t}\,[v]\,(x) + \int_t^T \left(P_{s-t}\,[\psi(\cdot, \nabla v\,(\cdot)\,G)]\,(x) - \lambda\right) ds. \tag{4.3}$$

Theorems 3.7 and 3.10 immediately yield existence and uniqueness of the mild solution of equation (4.1).

Theorem 4.1 *Assume that Hypotheses 2.1 and 3.1 hold.*

Then $(\bar{v}, \bar{\lambda})$ is a mild solution of the Hamilton-Jacobi-Bellman equation (4.1).

Conversely, if (v, λ) is a mild solution of (4.1) then, setting $Y_t^x = v(X_t^x)$ and $Z_t^x = \nabla v(X_t^x)G$, the triple (Y^x, Z^x, λ) is a solution of the EBSDE (3.1), which implies the uniqueness of mild solution in the sense that if $(\bar{v}^i, \bar{\lambda})$, $i = 1, 2$ are mild solutions of the Hamilton-Jacobi-Bellman equation (4.1) then $v^1(X_t^x) = v^2(X_t^x)$ and $\nabla v^1(X_t^x)G = \nabla v^2(X_t^x)G$ $\mathbb{P}$- a.s. for a.e. $t \geq 0$.

Proof. : The proof is identical to the one of Theorem 6.2 in [12].

5 Ergodic control

We fix a bounded function $F : H \to H$ and denote by X^x the solution of equation (2.1) with $\Upsilon = F$.

Assume that the Hypotheses 2.1 and 3.2 hold. Let U be a separable metric space. We define a control u as an $(\mathcal{F}_t)$-progressively measurable U-valued process. The cost corresponding to a given control is defined in the following way. We assume that the functions $R : U \to \Xi^*$ and $L : H \times U \to \mathbb{R}$ are measurable and satisfy, for some constant $c > 0$,

$$|R(u)| \leq c, \quad |L(x,u)| \leq c, \quad |L(x,u) - L(x',u)| \leq c\,|x - x'|, \qquad u \in U,\ x, x' \in H. \tag{5.1}$$

Given an arbitrary control u and $T > 0$, we introduce the Girsanov density

$$\rho_T^u = \exp\left(\int_0^T R(u_s)dW_s - \frac{1}{2}\int_0^T |R(u_s)|^2_{\Xi^*}\,ds\right)$$

and the probability $\mathbb{P}^u_T = \rho^u_T \mathbb{P}$ on $\mathcal{F}_T$. The ergodic cost corresponding to u and the starting point $x \in H$ is

$$J(x,u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{u,T} \int_0^T L(X^x_s, u_s)ds, \tag{5.2}$$

where $\mathbb{E}^{u,T}$ denotes expectation with respect to $\mathbb{P}^u_T$. We notice that $W^u_t = W_t - \int_0^t R(u_s)ds$ is a Wiener process on $[0,T]$ under $\mathbb{P}^u_T$ and that

$$dX^x_t = (AX^x_t + F(X^x_t))dt + G(dW^u_t + R(u_t)dt), \quad t \in [0,T]$$

and this justifies our formulation of the control problem. Our purpose is to minimize the cost over all controls.

To this purpose we first define the Hamiltonian in the usual way

$$\psi(x,z) = \inf_{u\in U}\{L(x,u) + zR(u)\}, \qquad x \in H,\, z \in \Xi^*, \tag{5.3}$$

and we remark that if, for all x, z, the infimum is attained in (5.3) then by the Filippov Theorem, see [17], there exists a measurable function $\gamma : H \times \Xi^* \to U$ such that

$$\psi(x,z) = l(x,\gamma(x,z)) + zR(\gamma(x,z)).$$

We notice that under the present assumptions ψ is a Lipschitz function and $\psi(\cdot,0)$ is bounded (here the fact that R depends only on u is used). So if we assume the Hypotheses 2.1 and 3.2, then in Theorem 3.7 we have constructed, for every $x \in H$, a triple

$$(\bar{Y}^x, \bar{Z}^x, \bar{\lambda}) = (\bar{v}(X^x), \bar{\zeta}(X^x), \bar{\lambda}) \tag{5.4}$$

solution to the EBSDE (3.1).

Theorem 5.1 *Assume that the Hypotheses 2.1 and 3.2 hold, and that (5.1) holds as well.*

Moreover suppose that, for some $x \in H$, a triple (Y, Z, λ) verifies $\mathbb{P}$-a.s. equation (3.1) for all $0 \le t \le T$, where Y is a progressively measurable continuous process, Z is a process in $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$ and $\lambda \in \mathbb{R}$. Finally assume that there exists $c_x > 0$ (that may depend on x) such that $\mathbb{P}$-a.s.

$$|Y_t| \le c_x(1 + |X^x_t|^2), \text{ for all } t \ge 0.$$

Then the following holds:

(i) For arbitrary control u we have $J(x,u) \ge \lambda = \bar{\lambda}$, and the equality holds if $L(X^x_t, u_t) + Z_t R(u_t) = \psi(X^x_t, Z_t)$, $\mathbb{P}$-a.s. for almost every t.

(ii) If the infimum is attained in (5.3) then the control $\bar{u}_t = \gamma(X^x_t, Z_t)$ verifies $J(x,\bar{u}) = \bar{\lambda}$.

In particular, for the solution (5.4) mentioned above, we have:

(iii) For arbitrary control u we have $J(x,u) = \bar{\lambda}$ if $L(X^x_t, u_t) + \bar{\zeta}(X^x_t)R(u_t) = \psi(X^x_t, \bar{\zeta}(X^x_t))$, $\mathbb{P}$-a.s. for almost every t.

(iv) If the infimum is attained in (5.3) then the control $\bar{u}_t = \gamma(X^x_t, \bar{\zeta}(X^x_t))$ verifies $J(x,\bar{u}) = \bar{\lambda}$.

Proof. : The proof is identical to the one of Theorem 7.1 in [12].

Example 5.1.1 *We consider here an ergodic optimal control when the state equation is a stochastic heat equation. The difference with respect to the same example in [12] is that, if we have non-degenerate noise, we do not need to assume that the non-linearity f is decreasing. Namely we consider the following state equation*

$$\begin{cases} d_t X^u(t,\xi) = \left[\frac{\partial^2}{\partial \xi^2} X^u(t,\xi) + f(\xi, X^u(t,\xi)) + r(\xi, u(t,\xi))\right] dt + \sigma(\xi)\dot{W}(t,\xi)\,dt, \\ X^u(t,0) = X^u(t,1) = 0, \\ X^u(t,\xi) = x_0(\xi), \end{cases} \tag{5.5}$$

where $\dot{W}(t,\xi)$ is a space-time white noise on $[0,\infty[\times[0,1]$.
We also introduce the cost functional

$$J(x,u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \int_0^1 l(\xi, X_s^u(\xi), u_s(\xi))\,d\xi\,ds. \tag{5.6}$$

An admissible control $u(t,\xi)$ is a predictable process $u : \Omega \times [0,\infty[\times[0,1]$.

Then (the reduction to the abstract infinite dimensional framework is as in [10] Section 5.1) the results of Theorem 5.1 can be applied under the following assumptions:

1. *$f : [0,1]\times\mathbb{R}\to\mathbb{R}$ is measurable and bounded and*

$$|f(\xi,\eta_1) - f(\xi,\eta_2)| \le c_f|\eta_2 - \eta_1|,$$

   for a suitable constant c_f, almost all $\xi\in[0,1]$, and all $\eta_1,\eta_2\in\mathbb{R}$. Moreover we assume that $f(\xi,\cdot)\in C^1(\mathbb{R})$ for a.a. $\xi\in[0,1]$.

2. *$\sigma : [0,1]\to\mathbb{R}$ is measurable and bounded. Moreover $c_\sigma \le |\sigma(\xi)|$, for a.a. $\xi\in[0,1]$ and a suitable constant $c_\sigma > 0$.*

3. *$r : [0,1]\times\mathbb{R}$ is measurable and bounded and, for a.a. $\xi\in[0,1]$, the map $r(\xi,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous.*

4. *$l : [0,1]\times\mathbb{R}^2$ is measurable and bounded and, for a.a. $\xi\in[0,1]$, the map $l(\xi,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}$ is continuous.*

5. *$x_0\in L^2([0,1])$.*

6 Appendix

6.1 Proof of the coupling estimates

Proof. We use a coupling argument. The precise result needed here cannot be found in the literature and we give the full proof for completeness. In particular, we show that the constant $\hat{c}$ and $\hat{\eta}$ in (2.4) depend only on the supremum of Υ. Our proof is a mix of arguments found in [7], [15], [16], [22] and [18]. Note that the more analytical method from [4] could also be used.

To prove that the laws of the solutions starting from different initial data get closer in variation, we first wait that these solutions enter a fixed ball. Then, we construct a coupling of solutions starting from initial data in this ball. Iterating this argument we obtain the result.

In this section, $\kappa_i,\ i = 0,1\ldots$ denotes a constant which depends only on γ, M, G and $L_0 = \sup_{x\in H}|F(x)|$.

Step 1: Let $Z_t = \int_0^t e^{A(t-s)} G dW(s)$ and $\rho = X^x - Z$. Thanks to Hypothesis 2.1, we obtain by taking the scalar product of the equation verified by ρ with ρ:

$$\frac{1}{2}\frac{d}{dt}|\rho|^2 + k|\rho|^2 \le L|\rho| \le \frac{L_0^2}{2k} + \frac{k}{2}|\rho|^2$$

with $L_0 = \sup_{x\in H} |\Upsilon(x)|$. By Gronwall's lemma

$$|\rho_t|^2 \le e^{-kt}(|x|^2 + \frac{L_0^2}{k^2}).$$

Moreover

$$\mathbb{E}(|Z_t|^2) = \int_0^t |e^{A(t-s)}G|^2_{L_2(H;H)} ds \le |G|^2_{\mathcal{L}(H)} \int_0^t |e^{As}|^2_{L_2(H;H)} ds.$$

It follows

$$\mathbb{E}|X_t^x|^2 \le 2(|x|^2 e^{-kt} + \kappa_1), \tag{6.1}$$

with a constant κ_1 depending only on γ, M, G and $L_0 = \sup_{x\in H} |F(x)|$ but independent of $t > 0$. By the Markov property:

$$\mathbb{E}(|X^x_{(k+1)T}|^2 | \mathcal{F}_{kT}) \le 2(|X^x_{kT}|^2 e^{-kT}) + \kappa_1, \ k \ge 0. \tag{6.2}$$

Let us define for $R \ge 0$

$$C_k = \{|X^x_{kT}|^2 \ge R\}, \quad B_k = \cap_{j=0}^k C_j.$$

By Chebychev's inequality

$$\mathbb{P}(C_{k+1}|\mathcal{F}_{kT}) \le \frac{2e^{-kT}}{R}|X^x_{kT}|^2 + \frac{\kappa_1}{R}. \tag{6.3}$$

Multiply (6.2) and (6.3) by $\mathbf{1}_{B_k}$ and take the expectation to obtain, since $\mathbf{1}_{B_{k+1}} \le \mathbf{1}_{B_k}$,

$$\begin{pmatrix} \mathbb{E}(|X^x_{(k+1)T}|^2 \mathbf{1}_{B_{k+1}}) \\ \\ \mathbb{P}(B_{k+1}) \end{pmatrix} \le A \begin{pmatrix} \mathbb{E}(|X^x_{kT}|^2 \mathbf{1}_{B_k}) \\ \\ \mathbb{P}(B_k) \end{pmatrix}$$

where

$$A = \begin{pmatrix} 2e^{-kT} & \kappa_1 \\ & \\ \dfrac{2e^{-kT}}{R} & \dfrac{\kappa_1}{R} \end{pmatrix}.$$

Let $R = 4\kappa_1$ and choose T such that $e^{-kT} = \frac{1}{8}$. The eigenvalues of A are 0 and $2e^{-kT} + \frac{1}{4} \le \frac{1}{8}$. We deduce that

$$\mathbb{P}(B_k) \le \kappa_2 \left(\frac{1}{8}\right)^k (1 + |x|^2),$$

with κ_2 depending only on κ_1. Defining

$$\tau = \inf\{kT; |X^x_{kT}|^2 \le R\},$$

it follows

$$\mathbb{P}(\tau \ge kT) \le \mathbb{P}(B_k) \le \kappa_2 \left(\frac{1}{8}\right)^k (1 + |x|^2).$$

Thus, for $\eta T < 2\ln 2$,

$$\mathbb{E}(e^{\eta\tau}) \leq \kappa_3(1+|x|^2). \tag{6.4}$$

Step 2: We construct a coupling for $x, y \in B_R$, the ball of center 0 and radius R, $x \neq y$. Let $T \geq 0$ to be chosen below, we denote by μ_1 the law of X^x and μ_2 the law of X^y on $[0,T]$. Set $\tilde{X}_t = X_t^y + \frac{T-t}{T} e^{At}(x-y)$ and denote by $\tilde{\mu_2}$ the law of $\tilde{X}$ on $[0,T]$. Then

$$d\tilde{X} = (A\tilde{X} + F(\tilde{X}))dt + d\tilde{W},$$

where $\tilde{W}_t = W_t - \int_0^t h(s)ds$, with $h(s) = F(\tilde{X}_s) - F(X_s^y) - \frac{1}{T}e^{At}(x-y)$. By Girsanov's formula, $\tilde{W}$ is a Wiener process under a new probability measure $\tilde{\mathbb{P}}$. Therefore, under $\tilde{\mathbb{P}}$, $\tilde{X}$ has the law μ_1 while under $\mathbb{P}$ it has the law $\tilde{\mu_2}$. Of course μ_1 and $\tilde{\mu_2}$ are equivalent. Since $|h(t)| \leq 2L_0 + 2\frac{R}{T}$, we deduce that

$$\int_H \left(\frac{d\tilde{\mu}_2}{d\mu_1}\right)^3 d\mu_1 \leq \kappa_4.$$

We need the following result (see for instance [16]).

Proposition 6.1 *Let (μ_1, μ_2) be two probability measures on a Banach space E then*

$$\|\mu_1 - \mu_2\|_{TV} = \min \mathbb{P}(Z_1 \neq Z_2)$$

where the minimum is taken on all coupling (Z_1, Z_2) of (μ_1, μ_2). Moreover, there exists a coupling which realizes the infimum. We say that it is a maximal coupling. It satisfies[1]

$$\mathbb{P}(Z_1 = Z_2,\ Z_1 \in \Gamma) = \mu_1 \wedge \mu_2(\Gamma),\ \Gamma \in \mathcal{B}(E).$$

Moreover, if μ_1 and μ_2 are equivalent and if

$$\int_E \left(\frac{d\mu_2}{d\mu_1}\right)^{p+1} d\mu_1 \leq C$$

for some $p > 1$ and $C > 1$ then

$$\mathbb{P}(Z_1 = Z_2) = \mu_1 \wedge \mu_2(E) \geq (1 - \frac{1}{p}) \left(\frac{1}{pC}\right)^{1/(p-1)}.$$

We deduce the existence of a coupling $(V^{1,x,y}, \tilde{V}^{2,x,y})$ of $(\mu_1, \tilde{\mu}_2)$ such that

$$\mathbb{P}(V^{1,x,y} = \tilde{V}^{2,x,y}) \geq \frac{1}{4\kappa_4}.$$

Clearly, $(V_t^{1,x,y}, V_t^{2,x,y} = \tilde{V}_t^{2,x,y} - \frac{T-t}{T}e^{At}(x-y))_{t\in[0,T]}$ is a coupling of (μ_1, μ_2) on $[0,T]$ and

$$\mathbb{P}(V_T^{1,x,y} = V_T^{2,x,y}) \geq \mathbb{P}(V^{1,x,y} = \tilde{V}^{2,x,y}) \geq \frac{1}{4\kappa_4}. \tag{6.5}$$

Step 3: We now construct a coupling for any initial data. For $x = y$, we set

$$(V_t^{1,x,x}, V_t^{2,x,x}) = (X_t^x, X_t^x),\ t \in [0,T].$$

[1] Recall that if μ_1, μ_2 are absolutely continuous with respect to a measure μ (for instance $\mu = \mu_1 + \mu_2$), we have

$$d(\mu_1 \wedge \mu_2) = (\tfrac{d\mu_1}{d\mu} \wedge \tfrac{d\mu_2}{d\mu})d\mu.$$

If x or y is not in B_R, we set

$$(V_t^{1,x,y}, V_t^{2,x,y}) = (X_t^x, \bar{X}_t^y),\ t \in [0,T],$$

where $\bar{X}^y$ is the solution of equation (2.3) driven by a Wiener process $\bar{W}$ independent of W. The coupling of the laws of X^x, X^y for $t \geq 0$ is defined recursively by the formula

$$V_{nT+t}^{i,x,y} = V_t^{i,V_{nT}^{1,x,y},V_{nT}^{1,x,y}},\ t \in [0,T], \quad i = 1,2.$$

We then define the following sequence of stopping times:

$$L_m = \inf\{l > L_{m-1}, V_{lT}^{1,x,y}, V_{lT}^{2,x,y} \in B_R\}$$

with $L_0 = 0$. Evidently, (6.4) can be generalized to two solutions and we have:

$$\mathbb{E}(e^{\eta L_1 T}) \leq \kappa_3(1 + |x|^2 + |y|^2)$$

and

$$\mathbb{E}(e^{\eta(L_{m+1}-L_m)T}|\mathcal{F}_{L_mT}) \leq \kappa_3(1 + |V_{L_mT}^{1,x,y}|^2 + |V_{L_mT}^{2,x,y}|^2).$$

It follows

$$\begin{aligned}\mathbb{E}(e^{\eta L_{m+1}T}) &\leq \kappa_3\mathbb{E}\left(e^{\eta L_mT}(1 + |V_{L_mT}^{1,x,y}|^2 + |V_{L_mT}^{2,x,y}|^2)\right)\\ &\leq \kappa_3(1+2R^2)\mathbb{E}\left(e^{\eta L_mT}\right)\end{aligned}$$

and

$$\mathbb{E}\left(e^{\eta L_mT}\right) \leq \kappa_3^l(1+2R^2)^{l-1}(1 + |x|^2 + |y|^2).$$

Set now

$$\ell_0 = \inf\{l,\ V_{(L_l+1)T}^{1,x,y} = V_{(L_l+1)T}^{2,x,y}\}.$$

Since $V_{L_lT}^{1,x,y} = V_{L_lT}^{2,x,y} \in B_R$, we have by (6.5)

$$\mathbb{P}(\ell_0 > l+1 \big| \ell_0 > l) \leq (1 - \frac{1}{4\kappa_4}).$$

Since $\mathbb{P}(\ell_0 > l+1) = \mathbb{P}(\ell_0 > l+1\big|\ell_0 > l)\mathbb{P}(\ell_0 > l)$, we obtain

$$\mathbb{P}(\ell_0 > l) \leq (1 - \frac{1}{4\kappa_4})^l.$$

Then for $\gamma \geq 0$

$$\begin{aligned}\mathbb{E}(e^{\gamma L_{\ell_0}T}) &\leq \sum_{l\geq 0}\mathbb{E}(e^{\gamma L_lT}\mathbf{1}_{l=\ell_0})\\ &\leq \sum_{l\geq 0}\mathbb{P}(l=\ell_0)^{1-\gamma/\eta}(\mathbb{E}(e^{\eta L_lT}))^{\gamma/\eta}\\ &\leq \sum_{l\geq 0}\left(1 - \frac{1}{4\kappa_4}\right)^{(l-1)(1-\gamma/\eta)}\ \left[\kappa_3^l(1+2R^2)^{l-1}(1+|x|^2+|y|^2)\right]^{\gamma/\eta}.\end{aligned}$$

We choose $\gamma \leq \eta$ such that

$$\left(1 - \frac{1}{4\kappa_4}\right)^{1-\gamma/\eta}\ \left[\kappa - 3(1+2R^2)\right]^{\gamma/\eta} < 1$$

and deduce

$$\mathbb{E}(e^{\gamma L_{\ell_0} T}) \leq \kappa_5(1+|x|^2+|y|^2).$$

Since

$$n_0 = \inf\{k,\ V_{kT}^{1,x,y} = V_{kT}^{2,x,y}\} \leq L_{\ell_0} + 1$$

it follows

$$\mathbb{E}(e^{\gamma n_0 T}) \leq \kappa_5(1+|x|^2+|y|^2)$$

and

$$\mathbb{P}(V_{kT}^{1,x,y} \neq V_{kT}^{2,x,y}) = \mathbb{P}(k \geq n_0) \leq \kappa_5(1+|x|^2+|y|^2)e^{-\gamma kT}.$$

Moreover

$$\begin{aligned}\mathbb{P}(V_{kT+t}^{1,x,y} \neq V_{kT+t}^{2,x,y}) \leq \mathbb{P}(V_{kT}^{1,x,y} \neq V_{kT}^{2,x,y}) &\leq \kappa_5(1+|x|^2+|y|^2)e^{-\gamma kT} \\ &\leq \kappa_6(1+|x|^2+|y|^2)e^{-\gamma(nT+t)}.\end{aligned}$$

We deduce for $\phi \in B_b(T^d)$ with $\sup_{x\in H}|\phi(x)| \leq 1$

$$\begin{aligned}|\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| &= \left|\mathbb{E}(\phi(V_t^{1,x,y}) - \phi(V_t^{2,x,y}))\right| \\ &\leq 2\kappa_6(1+|x|^2+|y|^2)e^{-\gamma t}.\end{aligned}$$

□

6.2 Proof of the recurrence

Our method consists in applying Proposition 3.4.5 in [6], and we apply Doob's Theorem (see Theorem 4.2.1 in [6]) in order to verify the conditions of Proposition 3.4.5 in [6].

Proof.

Let us first introduce an auxiliary Markovian semigroup $\mathcal{R}_t[\phi](x) = \mathbb{E}\phi(U_t^x)$ corresponding to the Markov process U where

$$U_t^x = e^{tA}x + \int_0^t e^{(t-s)A} G\, dW_s.$$

We denote

$$\mathcal{R}(t,x,\Gamma) = \mathbb{P}(U_t^x \in \Gamma) = \mathcal{R}_t[1_\Gamma](x), \qquad \Gamma \in \mathcal{B}(H),$$

the transition probabilities corresponding to U.

By Hypothesis 2.1, the Markovian semigroup $\mathcal{R}$ admits a unique invariant measure ν (see Theorem 6.3.3 in [6]). Moreover, $\mathcal{R}$ is irreducible (i.e., $\mathcal{R}(t,x,\Gamma) > 0$ for all Γ which is open and non empty set in H, see Theorem 7.2.1 in [6]), and t-regular for any $t > 0$ (i.e., the measures $\{\mathcal{R}(t,x,\cdot) : x \in H\}$ are equivalent, see Theorem 7.3.1 in [6]).

Next, recall that $\mathcal{P}_t[\phi](x) = \hat{\mathbb{E}}\phi(\hat{X}_t^x)$ (where $\hat{\mathbb{E}}$ means expectation with respect to probability $\hat{\mathbb{P}}$) is the Markovian semigroup corresponding to weak solutions $\hat{X}^x$ of the equation (2.3) (we notice that the solutions of equation (2.3) are unique in law). And we denote

$$\mathcal{P}(t,x,\Gamma) = \hat{\mathbb{P}}(\hat{X}_t^x \in \Gamma) = \mathcal{P}_t[1_\Gamma](x), \qquad \Gamma \in \mathcal{B}(H)$$

the transition probabilities corresponding to $\hat{X}$. From the Girsanov theorem, the semigroup $\mathcal{P}_t$ can be represented, for $t \leq T$, by

$$\mathcal{P}_t[\phi](x) = \mathbb{E}(\rho_T^x \phi(U_t^x))$$

where $\rho_T^x = \exp\Big(- \int_0^t \langle G^{-1}\Upsilon(U_t^x), dW_s \rangle_\Xi - \frac{1}{2}\int_0^t |G^{-1}\Upsilon(U_t^x)|_\Xi^2 ds \Big) > 0$, $\mathbb{P}$-a.s.

Consequently, $\mathcal{P}$ is irreducible (i.e., $\mathcal{P}(t,x,\Gamma) > 0$ for all Γ which is open and non empty set in H), and t-regular for any $t > 0$ (i.e., the measures $\{\mathcal{P}(t,x,\cdot) : x \in H\}$ are equivalent). And the above representation also implies that the semigroup $\mathcal{P}$ is stochastically continuous (i.e., $\lim_{t \searrow 0} \mathcal{P}_t[\phi](x) = \phi(x)$ for all $x \in H$ and $\phi \in \mathcal{C}_b(H)$).

Moreover, by a similar argument to that of Theorem 8.4.4 in [6], there exists a measurable function $\eta : H \to \mathbb{R}^+$ such that $\mu = \eta d\nu$ is the unique invariant measure corresponding to semigroup $(\mathcal{P}_t)_{t \geq 0}$. Indeed it is clear from the proof of Theorem 8.4.3 in [6] that Theorem 8.4.4 in [6] remains true whenever F can be approximated by a sequence of $\mathcal{C}_b^2$ functions converging in the bounded pointwise convergence sense.

We are now in position to apply Doob's Theorem (see Theorem 4.2.1 in [6]) to obtain that

- μ is strongly mixing and $\mathcal{P}_t(x,\Gamma) \to \mu(\Gamma)$ for all $\Gamma \in \mathcal{B}(H)$.
- μ is equivalent to all measures $\mathcal{P}_t(x,\cdot)$.

Let Γ be an open and non empty set in H. As μ is equivalent to all measures $\mathcal{P}_t(x,\cdot)$ and $\mathcal{P}$ is irreducible, $\mu(\Gamma) > 0$. Therefore, $\liminf_{t\to\infty} \mathcal{P}_t(x,\Gamma) \to \mu(\Gamma) > 0$. By Proposition 3.4.5 in [6], the Markovian semigroup $\mathcal{P}$ is recurrent and $\mathbb{P}\{\exists t > 0 : \hat{X}_t \in \Gamma\} = 1$.

In particular if, for all $T > 0$, $\hat{X}^x$ is a weak solution of equation (2.3) in [0,T], under the probability $\hat{\mathbb{P}}_T^x$, and $\tau = \inf\{t : |\hat{X}_t^x| < \epsilon\}$ then

$$\lim_{T\to\infty} \hat{\mathbb{P}}_T^x\{\tau^x < T\} = \lim_{T\to\infty} \hat{\mathbb{P}}_T^x\{\inf_{0\leq t\leq T} |\hat{X}_t^x| < \epsilon\} = \lim_{T\to\infty} \mathbb{P}\{\inf_{0\leq t\leq T} |\hat{X}_t^x| < \epsilon\} = 1. \ \square$$

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